THE ORNSTEIN-UHLENBECK SEMIGROUP IN EXTERIOR DOMAINS

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ABSTRACT. Let Ω be an exterior domain in $\mathbb{R}^n$. It is shown that Ornstein-Uhlenbeck operators L generate C_0-semigroups on $L^p(\Omega)$ for $p \in (1, \infty)$ provided $\partial\Omega$ is smooth. The method presented also allows to determine the domain $D(L)$ of L and to prove L^p-L^q smoothing properties of e^{tL}. If $\partial\Omega$ is only Lipschitz, results of this type are shown to be true for p close to 2.

1. Introduction

In recent years, many authors have considered Ornstein-Uhlenbeck operators (see [1], [2], [4], [8], [9], [10]), either from the point of view of analysis or stochastics. Note that all the articles deal with the realization of these operators in certain function spaces over the whole space $\mathbb{R}^n$, such as $C_b(\mathbb{R}^n)$, $\mathrm{BUC}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

On the other hand there is a considerable interest in such operators defined in exterior domains of $\mathbb{R}^n$, see e.g. [6], [5], [3]. Whereas the situation for $L^p(\mathbb{R}^n)$, equipped either with the Lebesgue measure or the invariant measure, can be regarded as fairly well understood, this is not the case for exterior domains.

In this note we consider Ornstein-Uhlenbeck operators L in domains Ω, where Ω denotes the exterior of a compact set K, often called the obstacle. We prove that L generates a C_0-semigroup T on $L^p(\Omega)$ for all $p \in (1, \infty)$. Making use of a result on iterated convolutions, we then show that, in addition, the semigroup T satisfies so-called L^p-L^p smoothing properties. Furthermore, our approach to Ornstein-Uhlenbeck operators allows us to determine precisely the domain $D(L)$ in $L^p(\Omega)$ as the set of all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $Mx \cdot \nabla u \in L^p(\Omega)$. Here M denotes an arbitrary $n \times n$-matrix having real entries.

If Ω is the exterior of a Lipschitz domain, the situation is quite different. In fact, as in the situation of the Dirichlet-Laplacian on Lipschitz domains, results of the above type only hold true for those values of p which are close to 2.

2. Ornstein-Uhlenbeck operators in exterior smooth domains

In this section we consider operators of the form

$$\mathcal{L}u(x) := \Delta u(x) + Mx \cdot \nabla u(x), \quad x \in \Omega$$

in $L^p(\Omega)$, where $1 < p < \infty$, $M \in \mathbb{R}^{n \times n} \setminus \{0\}$ and Ω is $\mathbb{R}^n$, a bounded domain with $C^{1,1}$-boundary or an exterior domain. The latter means that $\Omega = \mathbb{R}^n \setminus K$, where $K \subset \mathbb{R}^n$ denotes a compact set with $C^{1,1}$-boundary. We set

$$\begin{aligned} L_\Omega u &:= \mathcal{L}u \\ D(L_\Omega) &:= \{u \in W^{2,p}(\Omega) \cap W_0^1(\Omega) : Mx \cdot \nabla u \in L^p(\Omega)\}. \end{aligned}$$

Here and in the following we often write Mx instead of $M\cdot$ for the function $x \mapsto Mx$ for simplicity. We start by showing that L_Ω is quasi-dissipativ.

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Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be as above. Then the operator $\omega + L_\Omega$ is dissipative in $L^p(\Omega)$ provided $\omega > -\frac{\operatorname{tr} M}{p}$.*

Proof. Let $u \in D(L_\Omega),\ u^* = |u|^{p-2}\overline{u}$. Since Δ is dissipative we get

$$\operatorname{Re}\,(L_\Omega u, u^*) = \operatorname{Re}\,(\Delta u, u^*) + \operatorname{Re}\,(Mx \cdot \nabla u, u^*) \leq \operatorname{Re}\,(Mx \cdot \nabla u, u^*).$$

By a short formal calculation we see that

$$\begin{aligned}\operatorname{Re} \int_\Omega Mx \cdot \nabla u(x) u^*(x)\,\mathrm{d}x &= -\operatorname{Re} \int_\Omega u(x) \left(\sum_{i=1}^n \partial_i \left(\left(\sum_{k=1}^n m_{ik} x_k \right) u^*(x) \right) \right) \mathrm{d}x \\ &= -\operatorname{tr} M \int_\Omega u(x) u^*(x)\,\mathrm{d}x - \operatorname{Re} \int_\Omega u(x) Mx \cdot \nabla u^*(x)\,\mathrm{d}x \\ &= -\operatorname{tr} M \int_\Omega |u(x)|^p\,\mathrm{d}x - (p-1)\operatorname{Re} \int_\Omega (Mx \cdot \nabla u(x)) u^*(x)\,\mathrm{d}x.\end{aligned}$$

To make this calculation precise, we need to multiply by a smooth cut-off function and, in the case where $p < 2$, we additionally need to approximate $|u|^{p-2}$ by $(|u|^2 + \delta^2)^{p-2/2}$. In any case, we end up with

$$\operatorname{Re}\,(L_\Omega u, u^*) \leq -\frac{\operatorname{tr} M}{p} \|u\|^p_{L^p(\Omega)}.$$

□

We next state a result saying that $L_{\mathbb{R}^n}$ generates a C_0-semigroup on $L^p(\mathbb{R}^n)$ with growth bound $-\frac{\operatorname{tr} M}{p}$. Indeed, the following result was proved by Lunardi and Vespri [8], Metafune [9] and Metafune, Prüss, Rhandi and Schnaubelt [10].

Proposition 2.2. *Let $1 < p < \infty$. Then the operator $L_{\mathbb{R}^n}$ generates a C_0-semigroup $(e^{tL_{\mathbb{R}^n}})_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ satisfying $\|e^{tL_{\mathbb{R}^n}}\| \leq e^{-\frac{\operatorname{tr} M}{p} t}$, $t \geq 0$. Moreover, the semigroup $(e^{tL_{\mathbb{R}^n}})_{t\geq 0}$ is given by*

$$(e^{tL_{\mathbb{R}^n}} f)(x) = \frac{1}{(4\pi)^{n/2} (\det Q_t)^{1/2}} \int_{\mathbb{R}^n} f(e^{tM} x - y) e^{-\frac{1}{4}(Q_t^{-1} y,\, y)}\,\mathrm{d}y, \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.1}$$

where Q_t for $t > 0$ is given by $Q_t := \int_0^t e^{sM} e^{sM^T}\,\mathrm{d}s$.

In the following lemma we collect some mapping properties of $e^{tL_{\mathbb{R}^n}}$ and the resolvent $R(\lambda, L_{\mathbb{R}^n})$.

Lemma 2.3. *Let $1 < p < \infty$, $p \leq q < \infty$. Then the following holds.*

(a) *For $\lambda_0 > -\frac{\operatorname{tr} M}{p}$ there exists $C > 0$ such that*

$$\|\nabla(\lambda - L_{\mathbb{R}^n})^{-1} f\|_p \leq C \left(\lambda + \frac{\operatorname{tr} M}{p} \right)^{-\frac{1}{2}} \|f\|_p, \quad \lambda > \lambda_0,\ f \in L^p(\mathbb{R}^n).$$

(b) *There exist $C, \omega \geq 0$ such that*

$$\|e^{tL_{\mathbb{R}^n}} f\|_q \leq C t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \mathrm{e}^{\omega t} \|f\|_p, \quad t > 0,\ f \in L^p(\mathbb{R}^n).$$

(c) *$e^{tL_{\mathbb{R}^n}} f \in W^{m,p}(\mathbb{R}^n)$ for all $f \in L^p(\mathbb{R}^n)$, $m \geq 0$, $t > 0$ and there are constants $C_m, \omega_m \geq 0$, independent of f, such that*

$$\|D^\alpha e^{tL_{\mathbb{R}^n}} f\|_p \leq C_m t^{-\frac{m}{2}} e^{\omega_m t} \|f\|_p,\ t > 0,\ |\alpha| = m.$$

Proof. Since $L_{\mathbb{R}^n}$ generates a C_0-semigroup with growth bound $-\frac{\operatorname{tr} M}{p}$, we have

$$\|R(\lambda, L_{\mathbb{R}^n})\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq C\lambda^{-1}, \quad \lambda > -\frac{\operatorname{tr} M}{p}.$$

Moreover, for $\lambda_0 > -\frac{\operatorname{tr} M}{p}$ there exists $C > 0$ such that

$$\|R(\lambda, L_{\mathbb{R}^n})\|_{\mathcal{L}(L^p(\mathbb{R}^n), W^{2,p}(\mathbb{R}^n))} \leq C\|L_{\mathbb{R}^n} R(\lambda, L_{\mathbb{R}^n})\|_{\mathcal{L}(L^p(\mathbb{R}^n))} + \|R(\lambda, L_{\mathbb{R}^n})\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq C, \quad \lambda > \lambda_0.$$

Therefore, assertion (a) follows by interpolation. For a proof of assertions (b) and (c) see [5, Proposition 3.4]. □

We next consider the operator L_Ω where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{1,1}$-boundary $\partial\Omega$. By standard arguments, L_Ω generates an analytic C_0-semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$. In fact, the following lemma can be proved by using interpolation arguments and the Gagliardo-Nirenberg inequality.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$-boundary and $1 < p < \infty$, $p \leq q < \infty$. Then the operator L_Ω generates an analytic C_0-semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$. Moreover,*

(a) *for $\lambda_0 > -\frac{\operatorname{tr} M}{p}$ there exists a constant $C > 0$ such that*

$$\|\nabla(\lambda - L_\Omega)^{-1} f\|_p \leq C \left(\lambda + \frac{\operatorname{tr} M}{p}\right)^{-\frac{1}{2}} \|f\|_p, \quad \lambda > \lambda_0,\ f \in L^p(\Omega),$$

(b) *there exist $C, \omega \geq 0$ such that*

$$\|e^{tL_\Omega} f\|_q \leq C t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} e^{\omega t} \|f\|_p, \quad t > 0,\ f \in L^p(\Omega),$$

(c) *$e^{tL_\Omega} f \in W^{1,p}(\Omega)$ for all $f \in L^p(\Omega)$, $t > 0$ and there are constants $C, \omega \geq 0$, independent of f such that*

$$\|\nabla e^{tL_\Omega} f\|_p \leq C t^{-\frac{1}{2}} e^{\omega t} \|f\|_p,\ t > 0.$$

We now consider the operator L_Ω in exterior domains Ω with $C^{1,1}$-boundary and show that L_Ω generates a C_0-semigroup on $L^p(\Omega)$. The main result of this section reads as follows.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary and $1 < p < \infty$. Then the operator L_Ω generates a C_0-semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{-tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$.*

Proof. By the Lumer-Phillips theorem it suffices to show that $\omega + L_\Omega$ is dissipative for suitable $\omega \in \mathbb{R}$ and that $(\lambda - L_\Omega)D(L_\Omega) = L^p(\Omega)$ for some $\lambda > \omega$. Recall that dissipativity of L_Ω was already proved in Lemma 2.1.

In order to prove the range condition, consider the problem

$$\begin{aligned} \lambda u(x) - \Delta u(x) - Mx \cdot \nabla u(x) &= f(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega \end{aligned}$$

for $\lambda > -\frac{\operatorname{tr} M}{p}$. Choose $R > 0$ such that $K \subset \{x \in \mathbb{R}^n : |x| < R\}$ and set $D := \{x \in \Omega : |x| < R + 3\}$. Given $f \in L^p(\Omega)$, we define $f_0 \in L^p(\mathbb{R}^n)$ and $f_D \in L^p(D)$, respectively, by

$$f_0(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \qquad \text{and} \qquad f_D := f|_D.$$

Choose $\varphi \in C^\infty(\Omega)$, such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 0, & |x| \le R+1, \\ 1, & |x| \ge R+2. \end{cases}$$

Finally, we set

$$w_\lambda := \varphi u_\lambda + (1-\varphi)u_\lambda^D,$$

where $u_\lambda := (\lambda - L_{\mathbb{R}^n})^{-1} f_0$ and $u_\lambda^D := (\lambda - L_D)^{-1} f_D$. Since $\varphi \in C^\infty(\Omega)$ and $\nabla\varphi$ has compact support, we obviously have $w_\lambda \in D(L_\Omega)$. We calculate

$$\Delta w_\lambda = \varphi \Delta u_\lambda + (1-\varphi)\Delta u_\lambda^D + 2\nabla\varphi\nabla(u_\lambda - u_\lambda^D) + \Delta\varphi(u_\lambda - u_\lambda^D).$$

Therefore, given $f \in L^p(\Omega)$, w_λ satisfies

$$\lambda w_\lambda - \Delta w_\lambda - Mx \cdot \nabla w_\lambda = f + T(\lambda)f,$$

where

$$T(\lambda)f = -2\nabla\varphi \cdot \nabla(u_\lambda - u_\lambda^D) - [\Delta\varphi + (Mx \cdot \nabla\varphi)](u_\lambda - u_\lambda^D).$$

Note that, by Proposition 2.2, Lemma 2.3(a) and Lemma 2.4, for $\lambda_0 > -\frac{\operatorname{tr} M}{p}$ there exists $C > 0$, independent of $\lambda > \lambda_0$ and f, such that

$$\|\nabla(u_\lambda - u_\lambda^D)f\|_p \le C(\lambda + \frac{\operatorname{tr} M}{p})^{-\frac{1}{2}}\|f\|_p, \text{ and } \|u_\lambda - u_\lambda^D\|_p \le C(\lambda + \frac{\operatorname{tr} M}{p})^{-1}\|f\|_p.$$

We thus see that there is $\tilde\lambda_0 > 0$ such that $\|T(\lambda)\| < 1$ provided $\lambda > \tilde\lambda_0$. Hence, given $f \in L^p(\Omega)$ and $\lambda > \tilde\lambda_0$, set $U(\lambda)f := \varphi(\lambda - L_{\mathbb{R}^n})^{-1} f_0 + (1-\varphi)(\lambda - L_D)^{-1} f_D$. This shows that the function $V_\lambda := U(\lambda)(1 + T(\lambda))^{-1} f$ belongs to $D(L_\Omega)$ and $(\lambda - L_\Omega)V_\lambda = f$ for $\lambda > \tilde\lambda_0$. □

3. L^p-L^q Smoothing properties of e^{tL_Ω}

In the following we extend the smoothing properties of the semigroups defined on $\mathbb{R}^n$ and on bounded domains given in Lemma 2.3 and Lemma 2.4 to the case of exterior domains. More precisely, the following theorem holds.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ denote an exterior domain with $C^{1,1}$-boundary and let $1 < p < \infty$, $p \le q < \infty$. Then there exist $C, \omega > 0$ such that*

(a) $\|e^{tL_\Omega} f\|_q \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{\omega t}\|f\|_p, \quad t > 0,\ f \in L^p(\Omega)$,

(b) $\|\nabla e^{tL_\Omega} f\|_p \le Ct^{-\frac{1}{2}} e^{\omega t}\|f\|_p, \quad t > 0,\ f \in L^p(\Omega)$.

In order to prove Theorem 3.1 we make use of the following lemma on iterated convolutions proved in [3].

Lemma 3.2. *Let X, Y be Banach spaces and let $T : (0,\infty) \to \mathcal{L}(Y, X)$ and $S : (0,\infty) \to \mathcal{L}(Y)$ be strongly continuous functions. Assume that*

$$\|T(t)\|_{\mathcal{L}(Y,X)} \le C_0 t^\alpha e^{\omega t}, \quad \|S(t)\|_{\mathcal{L}(Y)} \le C_0 t^\beta e^{\omega t}, \quad t > 0,$$

for some $C_0, \omega > 0$ and $\alpha, \beta > -1$. For $f \in Y$ set $T_0(t)f := T(t)f$ and

$$T_n(t)f := \int_0^t T_{n-1}(t-s)S(s)f \, ds, \quad n \in \mathbb{N},\ t > 0.$$

Then there exist $C, \tilde{\omega} > 0$ *such that*

$$\sum_{n=0}^{\infty} \|T_n(t)f\|_X \leq Ct^{\alpha} \mathrm{e}^{\tilde{\omega}t} \|f\|_Y, \quad t > 0.$$

Proof of Theorem 3.1. Recall from the proof of Theorem 2.5 that the inverse of $(\lambda - L_\Omega)$ can be written as

$$(\lambda - L_\Omega)^{-1} f = U(\lambda)(1 + T(\lambda))^{-1} f, \quad f \in L^p(\Omega)$$

with $U(\lambda)$ and $T(\lambda)$ as above. Then the Laplace-Transforms of the strongly continous functions $S : [0, \infty) \to \mathcal{L}(L^p(\Omega))$ and $H : [0, \infty) \to \mathcal{L}(L^p(\Omega))$ defined by

$$S(t)f := \varphi \mathrm{e}^{tL_{\mathbb{R}^n}} f_0 + (1 - \varphi) \mathrm{e}^{tL_D} f_D,$$
$$H(t)f := -2\nabla\varphi \cdot \nabla(\mathrm{e}^{tL_{\mathbb{R}^n}} f_0 - \mathrm{e}^{tL_D} f_D) - [\Delta\varphi + (Mx \cdot \nabla\varphi)](\mathrm{e}^{tL_{\mathbb{R}^n}} f_0 - \mathrm{e}^{tL_D} f_D)$$

are given by $U(\lambda)$ and $T(\lambda)$, respectively. Clearly, there exist C, $\omega > 0$ such that

$$\|H(t)\|_{\mathcal{L}(L^p(\Omega))} \leq Ct^{-\frac{1}{2}} e^{\omega t}, \qquad \|S(t)\|_{\mathcal{L}(L^p(\Omega))} \leq Ce^{\omega t}, \qquad t > 0.$$

For $f \in L^p(\Omega)$ we set $T_0(t)f := S(t)f$ and define

$$T_n(t)f := \int_0^t T_{n-1}(t - s)H(s)f \, \mathrm{d}s, \quad n \in \mathbb{N}, \quad t > 0.$$

It then follows from Lemma 3.2 that

$$T_\Omega(t)f = \sum_{n=0}^{\infty} T_n(t)f$$

is well defined for all $t > 0$ and exponentially bounded. Thus, by Lebesgue's theorem and the convolution theorem for Laplace-Transforms,

$$\int_0^\infty e^{-\lambda t} T_\Omega(t) \mathrm{d}t = \sum_{n=0}^{\infty} \int_0^\infty e^{-\lambda t} \, T_n(t) \mathrm{d}t = \sum_{n=0}^{\infty} U(\lambda) T(\lambda)^n = (\lambda - L_\Omega)^{-1}$$

for λ large enough and hence $T_\Omega(t) = \mathrm{e}^{tL_\Omega}$ for $t \geq 0$. Now, for $0 \leq \frac{1}{p} - \frac{1}{q} < \frac{2}{n}$, assertion (a) follows by Lemma 3.2, Lemma 2.3(b) and Lemma 2.4(b). Iterating this argument yields (a) for $1 < p \leq q < \infty$. Assertion (b) follows in a similar way. □

4. The Ornstein-Uhlenbeck operator in Lipschitz domains and Lipschitz domains satisfying a uniform outer ball condition

In this section let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain or an exterior Lipschitz domain. We study the operator $L_\Omega u := \mathcal{L}u$ with domain

$$D(L_\Omega) = \{u \in W_0^{1,p}(\Omega) : \Delta u \in L^p(\Omega),\ Mx \cdot \nabla u \in L^p(\Omega)\},$$

where p lies in some interval around 2.

It follows from [11, Theorem 3.2.1] that for bounded Lipschitz domains the Dirichlet-Laplacian with domain consisting of all $u \in W_0^{1,p}(\Omega)$ such that $\Delta u \in L^p(\Omega)$ generates an analytic C_0-semigroup on $L^p(\Omega)$ for $(3 + \varepsilon)' < p < 3 + \varepsilon$ where $\varepsilon > 0$ depends on the domain Ω and $(3 + \varepsilon)'$ denotes the conjugate exponent to $3 + \varepsilon$. By interpolation of the resolvent estimate for the generator of an analytic semigroup $\|u\|_p \leq C\lambda^{-1} \|(\lambda - \Delta)u\|_p$ and the estimate $\|u\|_{1+\delta,p} \leq C \|\Delta u\|_p$ for some small $\delta > 0$ from [7, Theorem 1.1], we get the gradient estimate

$$\|u\|_{1,p} \leq C|\lambda|^{-\Theta} \|\Delta u\|_p + C|\lambda|^{1-\Theta} \|u\|_p, \quad u \in D(L_\Omega),\ \lambda \in \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\},$$

where $\Theta = \delta/(1+\delta)$ for $(3+\varepsilon)' < p < 3+\varepsilon$. The next lemma follows from this result by perturbation theory.

Lemma 4.1. *Let Ω be a bounded Lipschitz domain and $(3+\varepsilon)' < p < 3+\varepsilon$ where $\varepsilon > 0$ depends on Ω. Then the operator L_Ω generates an analytic semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$.*

Next, we turn to the case of exterior Lipschitz domains. We have the following result.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ denote an exterior domain with Lipschitz boundary. Let $2n/(n+1) - \delta < p < 2n/(n-1) + \delta$ where $\delta > 0$ depends on Ω. Then the operator L_Ω generates a C_0-semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$.*

Proof. Proceed in exactly the same way as in the proof of Theorem 2.5 noting that $u_\lambda^D \in D(L_D)$ and $w_\lambda \in D(L_\Omega)$. □

In the remaining part of this section we consider Ornstein-Uhlenbeck operators in a special class of Lipschitz domains Ω, namely domains satisfying a uniform outer ball condition. The latter means that there exists $R > 0$ such that for any $x \in \partial\Omega$ there exists an open ball $B \subseteq \Omega^c$ with radius R and $x \in \partial B$. Examples of such domains are the exterior of a propeller, a heart or a pac-man. For the rest of this section, we define the realization of $\mathcal{L}$ in $L^p(\Omega)$ for $1 < p \leq 2$ by $L_\Omega u := \mathcal{L}u$ and

$$D(L_\Omega) = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : Mx \cdot \nabla u \in L^p(\Omega)\}, \quad 1 < p \leq 2.$$

First, let Ω be a bounded Lipschitz domain satisfying a uniform outer ball condition. By [11, Theorem 3.2.6], the Laplacian with domain $D(\Delta) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ generates an analytic semigroup on $L^p(\Omega)$ for $1 < p \leq 2$. Using perturbation theory again, we can extend this result for the Ornstein-Uhlenbeck operator. Indeed, if Ω is a bounded Lipschitz domain satisfying a uniform outer ball condition, the operator L_Ω generates an analytic semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$. Finally, let $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain satisfying a uniform outer ball condition, where the latter means that Ω is a Lipschitz domain satisfying a uniform outer ball condition and Ω is the complement of some compact set. Then similarly as in Section 2 we obtain the following theorem.

Theorem 4.3. *Let $1 < p \leq 2$ and let $\Omega \subset \mathbb{R}^n$ be an exterior Lipschitz domain satisfying a uniform outer ball condition. Then the operator L_Ω generates a C_0-semigroup $(e^{tL_\Omega})_{t\geq 0}$ on $L^p(\Omega)$ satisfying $\|e^{tL_\Omega}\| \leq e^{-\frac{\operatorname{tr} M}{p}t}$, $t \geq 0$. Moreover, let $p \leq q \leq 2$. Then there exist $C, \omega > 0$ such that*

(a) $\|e^{tL_\Omega} f\|_q \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{\omega t} \|f\|_p, \quad t > 0,\ f \in L^p(\Omega),$
(b) $\|\nabla e^{tL_\Omega} f\|_p \leq Ct^{-\frac{1}{2}} e^{\omega t} \|f\|_p, \quad t > 0,\ f \in L^p(\Omega).$

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