On quasi-symmetric designs with intersection difference three

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Abstract

In a recent paper, Pawale [22] investigated quasi-symmetric 2-(v, k, λ) designs with intersection numbers $x > 0$ and $y = x + 2$ with $\lambda > 1$ and showed that under these conditions either $\lambda = x + 1$ or $\lambda = x + 2$, or $\mathcal{D}$ is a design with parameters given in the form of an explicit table, or the complement of one of these designs. In this paper, quasi-symmetric designs with $y - x = 3$ are investigated. It is shown that such a design or its complement has parameter set which is one of finitely many which are listed explicitly or $\lambda \leq x + 4$ or $0 \leq x \leq 1$ or the pair (λ, x) is one of $(7, 2)$, $(8, 2)$, $(9, 2)$, $(10, 2)$, $(8, 3)$, $(9, 3)$, $(9, 4)$ and $(10, 5)$. It is also shown that there are no triangle-free quasi-symmetric designs with positive intersection numbers x and y with $y = x + 3$.

1 Introduction

Let $\mathcal{D}$ be a 2-(v, k, λ) design. Here as usual, v denotes the number of points of $\mathcal{D}$, k the block size and λ the number of occurrences of pairs of points in the blocks of $\mathcal{D}$.

Then each point occurs in a constant number r of blocks of $\mathcal{D}$. If b denotes the number of blocks of $\mathcal{D}$, then the parameters (v, k, λ, r, b) satisfy the basic relations $bk = vr$, $\lambda(v-1) = r(k-1)$, and Fisher's inequality $b \geq v$.

For general notation and concepts in design theory, we refer to Beth, Jungnickel, and Lenz [2] or Hughes and Piper [7]. A design with $v = b$ (equivalently $r = k$) is known as a *symmetric* 2-(v, k, λ)-design. The *intersection numbers* of 2-(v, k, λ)-design are the cardinalities of the intersection of any two distinct blocks. It is well known that a 2-(v, k, λ)-design is symmetric if and only if $\mathcal{D}$ has exactly one intersection number (=λ). Let x and y be non-negative integers with $x \leq y < k$. A design $\mathcal{D}$ is called *quasi-symmetric* with intersection numbers x and y if any two distinct blocks of $\mathcal{D}$ intersect in x or y points and both intersection numbers are realized. We refer to Shrikhande and Sane [26] as a basic reference on quasi-symmetric designs. A quasi-symmetric design is called *proper* if $x \neq y$ and *improper* otherwise. Clearly symmetric designs are improper quasi-symmetric designs and any 2-$(v, k, 1)$ design with $b > v$ is a proper quasi-symmetric design with $x = 0$ and $y = 1$. Thus *linear spaces*, that is 2-$(v, k, 1)$ designs, give examples of proper and improper quasi-symmetric designs.

A 2-(v, k, λ) design is called *resolvable* if its blocks can be partitioned in subsets called *parallel classes* such that each parallel class partitions the point set. A partition of the blocks is called a *parallelism* with blocks in the same class being parallel. Two distinct parallel blocks are disjoint. If further, any two blocks from different parallel classes intersect in a constant number y (say) of points, the design is called *affine*. Affine designs are thus quasi-symmetric with $x = 0$ and y.

Examples of quasi-symmetric designs which are not symmetric, or affine designs, or linear spaces are rather rare, so construction methods of quasi-symmetric designs are of interest. The problem of classifying quasi-symmetric 2-designs, even for the case $x = 0$ appears to be a difficult open problem. As a consequence, one approach in the study of such designs has been to put additional parametric or structural restrictions. Baartmans and Shrikhande [1]; Limaye, Sane, and Shrikhande [11]; Mavron and Shrikhande [13] ; Cameron [6]; Sane and Shrikhande [24]; McDonough and Mavron [16]; Mavron, McDonough and Shrikhande [14] are some papers where additional structural conditions are imposed. Pawale [21] studies quasi-symmetric 2-designs satisfying a parametric condition of the form $y - x$ has a fixed value.

In a recent preprint, Pawale [22] obtained a parametric classification of proper quasi-symmetric 2-designs with $y - x = 2$, with $x > 0$ and $\lambda > 1$. It is shown in [22] that if $\mathcal{D}$ is a quasi-symmetric 2-design with these conditions, then either $\lambda = x + 1$ or $\lambda = x + 2$, or $\mathcal{D}$ is a design with parameters given in the form of an explicit table, or the complement of one of these designs.

Suppose now that $y = x + 3$ in a proper quasi-symmetric 2-design. The following are the currently known examples of such designs: The affine $2-(27, 9, 4)$ designs considered by Lam and Tonchev [10] with $x = 0$ and $y = 3$; the geometric 2-$(121, 13, 13)$ design $\mathcal{D} = PG_2(4, 3)$ of points and planes of the projective space $PG(4, 3)$, with $x = 1$ and $y = 4$; a non-geometric 2-$(121, 13, 13)$ design with $x = 1$, $y = 4$ given in Jungnickel and Tonchev [8], which is a special case of an infinite class of quasi-symmetric designs with $x = 1, y = q + 1$, where q is a prime power, with parameters the same as those of the geometric design $PG_d(2d, q)$; a class of quasi-symmetric 2-$(66, 30, 29)$ designs with $x = 12$ and $y = 15$ and a class of quasi-symmetric 2-$(78, 36, 30)$ designs with $x = 15$ and $y = 18$ constructed by Bracken, McGuire and Ward [3]. See also McDonough, Mavron and Ward [17] for an alternative description of quasi-symmetric designs with these parameters.

In this paper, we investigate quasi-symmetric 2-designs with $y-x = 3$, with a goal towards obtaining parametric classification of such designs. Calculation was greatly facilitated by the computer algebra system Maxima [15]. Section 2 contains preliminary results needed. The main results, which are to be found in Section 3 and Section 4, are the following:

Theorem 3.1. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x + 3$. If $x \geq 31$, then $\lambda \leq x + 14$.*

Theorem 3.2. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x + 3$. If $x \geq 28$ and $\lambda \geq x + 5$, then*

$$k + 1 \leq r < \frac{9}{2} + \frac{\lambda(2k^2(x + 1) - x(x + 3))}{2(k - 1)x(x + 3)}.$$

Theorem 3.3. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x + 3$. Then either $\lambda \leq x + 4$ or $x \leq 30$ or the set of parameters of $\mathcal{D}$ is one of the six listed in Table 1.*

Theorem 3.4. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters with $2 \leq x \leq 30$ and $y = x + 3$. Then the set of parameters of $\mathcal{D}$ is one of the 43 sets listed on Tables 3 and 4 or (λ, x) is one of the "exceptional" pairs $(7, 2)$, $(8, 2)$, $(9, 2)$, $(10, 2)$, $(8, 3)$, $(9, 3)$, $(9, 4)$ and $(10, 5)$.*

Theorem 4.6. *There are no proper triangle-free quasi-symmetric designs with non-zero intersection numbers x and y with $y = x + 3$.*

2 Preliminaries

Throughout this paper, we consider a proper quasi-symmetric design with *standard parameters* $(v, k, \lambda, r, b, x, y, a)$. This means that v is the number of points, b is the number of blocks, k is the number of points on a block, r is the number of blocks on a point, λ is the number of blocks on a pair of points, $\lambda > 1$, x and y are the sizes of intersections of pairs of blocks, each of x and y occurs as the size of a block intersection, $0 \le x < y < k$ and a is the number of blocks intersecting a given block in a set of size y. The complementary design of a quasi-symmetric design with parameters $(v, k, \lambda, r, b, x, y, a)$ is a quasi-symmetric design with parameters $(v, v-k, b-2r+\lambda, b-r, b, v-2k+x, v-2k+y, a)$. Consequently, either a design or its complement will have a block size which is no more than half of the size of the set of points. We may also refer, more briefly, to the quintuple (v, k, λ, r, b) as the set of design parameters. It will be convenient to assume that $2k \le v$.

Lemma 2.1. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Then (i) $vr = bk$, (ii) $(v-1)\lambda = (k-1)r$, (iii) $ay+(b-1-a)x = k(r-1)$, (iv) $ay(y-1)+(b-1-a)x(x-1) = k(k-1)(\lambda-1)$, (v) $(b-1)xy - k(r-1)(x+y-1)+k(k-1)(\lambda-1) = 0$, (vi) $(k-1)r^2 + \lambda r - bk\lambda = 0$, (vii) $y-x$ is a factor of $k-x$ and $r-\lambda$.*

Proof. (vi) $(k-1)r^2 = (v-1)\lambda r = vr\lambda - \lambda r = bk\lambda - \lambda r$. The remaining parts are well known and their proofs can be found in [26]. □

Using Lemma 2.1 (iii), (iv) and (v), we get the following lemma.

Lemma 2.2. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Then $a = \dfrac{k((k-1)(\lambda-1)-(r-1)(x-1))}{y(y-x)}$, $b-1 = \dfrac{k((r-1)(x+y-1)-(k-1)(\lambda-1))}{xy}$ and $b-1-a = \dfrac{k((r-1)(y-1)-(k-1)(\lambda-1))}{x(y-x)}$ if $x > 0$ and $a = \dfrac{k(r-1)}{y}$ if $x = 0$. Moreover, $a < b-1$.*

Lemma 2.3. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Then (i) $b > v$, (ii) $r > k$, (iii) if $\lambda > 1$ then $x < k^2/v < \lambda$, (iv) $2x < k$, (v) if $k \ge 4$ then $b \le v(v-1)/2$, (vi) if $k \ge 4$ then $\lambda \le k(k-1)/2$.*

Proof. (iii) From Lemma 2.1 (iii), $a = (k(r-1) - x(b-1))/(y-x)$. Since $\lambda \ge 2$, $y \ge 2$ and $a \ge 1$. Hence, $k(r-1) > x(b-1)$. So, $x < k(r-1)/(b-1) < kr/b = k^2r/bk = k^2r/vr = k^2/v$. Since $r > k$, $\lambda(v-1) = r(k-1) \ge k^2-1$. Hence, $\lambda v \ge k^2 - 1 + \lambda > k^2$, as $\lambda \ge 2$.

(iv) Since $v \ge 2k$, $k/v \le 1/2$. From (iii), $x < k/2$.

(v) Since $y < k$, there are no repeated blocks. Now apply [26, Theorem 3.16].

(vi) From (v), $br \le vr(v-1)/2 = bk(v-1)/2 = bk(k-1)r/2\lambda$. Hence, $\lambda \le k(k-1)/2$. □

Lemma 2.4. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Then $a_1r^2 + b_1r + c_1 = 0$, where $a_1 = (k-1)xy$, $b_1 = \lambda(xy - k^2(x+y-1))$, and $c_1 = \lambda k(k^2(\lambda-1) + k(x+y-\lambda) - xy)$.*

Proof. From Lemma 2.1 (vi), $(k-1)r^2+\lambda r-bk\lambda = 0$. Hence, $(k-1)xyr^2+\lambda xyr-bxyk\lambda = 0$. From Lemma 2.1 (v), $bxy = xy + k(r-1)(x+y-1) - k(k-1)(\lambda-1)$. Substituting, we get $(k-1)xyr^2 + \lambda xyr - (xy + k(r-1)(x+y-1) - k(k-1)(\lambda-1))k\lambda = 0$. Hence, $(k-1)xyr^2 + (xy - k^2(x+y-1))\lambda r - (xy - k(x+y-1) - k(k-1)(\lambda-1))k\lambda = 0$. Since $k(x+y-1) = k(x+y-\lambda) + k(\lambda-1)$, the result follows. $\square$

Calderbank [4, 5] has developed some very useful necessary conditions for the existence of a quasi-symmetric design on its parameters. We list three of these in the following theorems.

Theorem 2.5 (Calderbank [5, Theorem 2]). *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Then (a) $k(v-k)(k(v-k)-1)+(v-2)(v-1)(k-x)(k-y)-k(v-2)(v-k)(2k-x-y) \geq 0$ and (b) $k(v-6)(v-3)(v-k)(2k-x-y)2-2k(v-3)(v-k)(2k(v-k)-3v)(2k-x-y)+(6-v)(v-3)(v-1)(k-x)(k-y)(2k-x-y)+k(v-k)(5v+3k(v-k)(k(v-k)-2(v-1))-3)+(v-3)(k(v-k)(3v+2)-6(v-1)v)(k-x)(k-y) \geq 0$.*

Theorem 2.6 (Calderbank [4, Theorems 1 and 2]). *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters. Let $s_1, \ldots, s_l$ be the distinct intersection sizes of pairs of blocks. Let p be a prime. Suppose that $s_1 \equiv \ldots \equiv s_l \equiv s \pmod{p}$.*

If $p = 2$ then either

- *(a)* $r \equiv \lambda \pmod 4$, *or*
- *(b)* $s \equiv 0 \pmod 2$, $k \equiv 0 \pmod 4$ *and* $v \equiv \pm 1 \pmod 8$, *or*
- *(c)* $s \equiv 1 \pmod 2$, $k \equiv v \pmod 4$ *and* $v \equiv \pm 1 \pmod 8$.

If $p > 2$ then either

- *(a)* $r \equiv \lambda \pmod{p^2}$, *or*
- *(b)* $v \equiv 0 \pmod 2$, $v \equiv k \equiv s \equiv 0 \pmod p$ *and* $(-1)^{v/2}$ *is a square* $\pmod p$, *or*
- *(c)* $v \equiv 1 \pmod 2$, $v \equiv k \equiv s \not\equiv 0 \pmod p$ *and* $(-1)^{(v-1)/2}s$ *is a square* $\pmod p$, *or*
- *(d)* $r \equiv \lambda \equiv 0 \pmod p$ *and either*
  - *(i)* $v \equiv 0 \pmod 2$ *and* $v \equiv k \equiv s \not\equiv 0 \pmod p$, *or*
  - *(ii)* $v \equiv 0 \pmod 2$, $k \equiv s \equiv 0 \pmod p$ *and* vs *is a square* $\pmod p$, *or*
  - *(iii)* $v \equiv 1 \pmod{2p}$, $r \equiv 0 \pmod{p^2}$ *and* $k \equiv s \not\equiv 0 \pmod p$, *or*
  - *(iv)* $v \equiv p \pmod{2p}$, $r \equiv 0 \pmod{p^2}$ *and* $k \equiv s \equiv 0 \pmod p$, *or*
  - *(v)* $v \equiv 1 \pmod 2$, $k \equiv s \equiv 0 \pmod p$ *and* v *is not a square* $\pmod p$, *or*
  - *(vi)* $v \equiv 1 \pmod 2$, $k \equiv s \equiv 0 \pmod p$ *and* v *and* $(-1)^{(v-1)/2}$ *are squares* $\pmod p$.

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From Lemma 2.3, $\lambda \geq x+1$ and $k \geq 2x+1$.

From Lemma 2.4, $r = \dfrac{-b_1 \pm \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$. So, $r \leq \dfrac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$. Moreover, the discriminant $\Delta = b_1^2 - 4a_1c_1$ must be a perfect square. Letting $y = x+3$ and $\lambda = x+p$, we get Δ as a polynomial function of k, x and p, with degree 4 in k. We will show that if x and p are sufficiently large then Δ is negative; so, a lower bound on x will imply an upper bound on p. We will also show that if x and p are sufficiently large and then r is suitably bounded above.

Theorem 3.1. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x+3$. If $x \geq 31$ then $\lambda \leq x+14$.*

Proof. In this case, $\Delta = \lambda(\lambda F(k,x) + G(k,x))$, where $F(k,x) = 4k^4(1-x) + 8k^3x(x+3) - 4k^2x(x+2)(x+3) + x^2(x+3)^2$ and $G(k,x) = 4k(k-1)(k-x)(k-x-3)x(x+3)$.

Throughout the proof we assume that $x \geq 31$. We will show that if $\lambda \geq x+15$ then $\Delta < 0$.

Let $F^{(i)}(k,x) = \dfrac{\partial^i F(k,x)}{\partial k^i}$. Then $F^{(4)}(k,x) = 96(1-x) < 0$ since $x \geq 31$. Hence, $F^{(3)}(k,x)$ is decreasing as k increases.

We write $x = 31 + \zeta$. Then $F^{(3)}(2x+1,x) = -48(\zeta+29)(3\zeta+94) < 0$ for $\zeta \geq 0$. So $F^{(3)}(k,x) < 0$ for $k \geq 2x+1$. Noting that

$$\begin{aligned} F^{(2)}(2x+1,x) &= -8(13\zeta^3 + 1172\zeta^2 + 35155\zeta + 350790), \\ F^{(1)}(2x+1,x) &= -8(2\zeta+63)(3\zeta^3 + 263\zeta^2 + 7648\zeta + 73716), \\ F(2x+1,x) &= -16\zeta^5 - 2351\zeta^4 - 137570\zeta^3 - 4004595\zeta^2 - 57943436\zeta - 333047132, \end{aligned}$$

and repeatedly applying the preceding argument, we get $F(k,x) < 0$ for $k \geq 2x+1$. Hence, for $x \geq 31$ and $k \geq 2x+1$, $\lambda F(k,x) + G(k,x)$ decreases as λ increases. We will show that $\lambda F(k,x) + G(k,x) < 0$ when $\lambda = x+15$, and hence $\lambda F(k,x) + G(k,x) < 0$ when $\lambda \geq x+15$.

We now consider the case in which $\lambda = x+15$. Let $H(k,x) = (x+15)F(k,x) + G(k,x)$. Denoting partial derivatives with respect to k as before, we find $H^{(4)}(k,x) = -96(11x-15) < 0$ when $x \geq 2$.

$$\begin{aligned} H^{(3)}(2x+1,x) &= -48(31\zeta^2 + 1845\zeta + 27374). \\ H^{(2)}(2x+1,x) &= -24(40\zeta^3 + 3527\zeta^2 + 103205\zeta + 1001518). \\ H^{(1)}(2x+1,x) &= -4(89\zeta^4 + 10118\zeta^3 + 426789\zeta^2 + 7895932\zeta + 53861572). \\ H(2x+1,x) &= -71\zeta^5 - 8940\zeta^4 - 423367\zeta^3 - 8975086\zeta^2 - 73364052\zeta - 38146184. \end{aligned}$$

So, for $\zeta \geq 0$, we get $H^{(3)}(2x+1,x) < 0$, $H^{(2)}(2x+1,x) < 0$, $H^{(1)}(2x+1,x) < 0$ and $H(2x+1,x) < 0$. By an argument used earlier, $H(k,x) \leq H(2x+1,x) < 0$ for all $k \geq 2x+1$. Thus, when $x \geq 31$ and $\lambda \geq x+15$, $\Delta < 0$. Since this is impossible, $\lambda < x+15$ when $x \geq 31$. □

We should mention that the above theorem is a particular case for $y = x+3$ of a general result of Pawale [21], with an improved upper bound. Theorem 3.1 of [21] states: Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameter set $(v, b, r, k, \lambda; x, y)$ with $v \geq 2k$ and $z = y - x$. If $x \geq 1 + z + z^3$, then $x < \lambda < x + 1 + z + z^3$.

Theorem 3.2. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x+3$. If $x \geq 28$ and $\lambda \geq x+5$, then*

$$k+1 \leq r < \frac{9}{2} + \frac{\lambda(2k^2(x+1) - x(x+3))}{2(k-1)x(x+3)}$$

Proof. We consider the function $K(k,x,\lambda) = \Delta - 81x^2(x+3)^2(k-1)^2$. We denote derivatives as in Theorem 3.1. Then $K^{(4)}(k,x,\lambda) = -96\lambda((\lambda-x-5)(x-1)+x-5) < 0$ when $x \geq 6$ and $\lambda \geq x+5$; so $K^{(3)}(k,x,\lambda)$ is decreasing as k increases.

We will now compute $K^{(i)}(2x+1,x,\lambda)$ for $i = 0, \ldots, 3$ and we will observe that they are all negative. Thus $K^{(i)}(2x+1,x,\lambda) < 0$ for all $k \geq 2x+1$ when $x \geq 28$ and $\lambda \geq x+5$.

Replacing x by $28+\zeta$ and λ by $33+\eta+\zeta$, we get

$$\begin{aligned}
K^{(3)}(2x+1,x,\lambda) &= -48(\zeta+\eta+33)(3\eta\zeta^2+\zeta^2+163\eta\zeta+29\zeta+2210\eta+18), \\
K^{(2)}(2x+1,x,\lambda) &= -2((52\eta+41)\zeta^4+(52\eta^2+5896\eta+4042)\zeta^3+(4220\eta^2+248960\eta+ \\
&\quad 142625)\zeta^2+(113896\eta^2+4639172\eta+2077284)\zeta+(1022088\eta^2+ \\
&\quad 32178816\eta+9874440)), \\
K^{(1)}(2x+1,x,\lambda) &= -4((12\eta+50)\zeta^5+(12\eta^2+1651\eta+6993)\zeta^4+(1286\eta^2+90136\eta+ \\
&\quad 389924)\zeta^3+(51508\eta^2+2438663\eta+10832761)\zeta^2+(913446\eta^2+ \\
&\quad 32656642\eta+149910052)\zeta+(6048612\eta^2+172866976\eta+ \\
&\quad 826437372)), \\
K(2x+1,x,\lambda) &= -((16\eta+235)\zeta^6+(16\eta^2+2550\eta+40204)\zeta^5+(2111\eta^2+ \\
&\quad 167226\eta+2863058)\zeta^4+(110798\eta^2+5759930\eta+108632296)\zeta^3+ \\
&\quad (2889099\eta^2+109460478\eta+2316204943)\zeta^2+(37382828\eta^2+ \\
&\quad 1081543056\eta+26312182500)\zeta+(191730332\eta^2+4297903416\eta+ \\
&\quad 124418231964)).
\end{aligned}$$

We have established that $\Delta < (9x(x+3)(k-1))^2$ when $x \geq 28$ and $\lambda \geq x+5$. From Lemma 2.4, $r = (-b_1 \pm \sqrt{\Delta})/2a_1 \leq (-b_1 + \sqrt{\Delta})/2a_1 < (-b_1 + 9x(x+3)(k-1))/2a_1$. Since $a_1 = (k-1)x(x+3)$ and $b_1 = \lambda(x(x+3) - 2k^2(x+1))$, $r < 9/2 + \lambda(2k^2(x+1) - x(x+3))/(2(k-1)x(x+3))$. □

Theorem 3.3. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters and let $y = x + 3$. Then either $\lambda \leq x + 4$ or $x \leq 30$ or the set of parameters of $\mathcal{D}$ is one of the six listed in Table 1.*

Proof. Assume that $x \geq 31$ and $\lambda \geq x + 5$. By Theorem 3.1, $\lambda \leq x + 14$.

$\Delta = \lambda L(k, x, \lambda)$ where $L(k, x, \lambda) = \lambda F(k, x) + G(k, x)$ and $F(k, x)$ and $G(k, x)$ are as in the proof of Theorem 3.1. Then $L(k, x, \lambda) = 4k^4(x(x + 3) + \lambda(1 - x)) + 8k^3x(x + 3)(\lambda - x - 2) - 4k^2x(x + 3)(\lambda(x + 2) - x^2 - 5x - 3) - 4kx^2(x + 3)^2 + \lambda x^2(x + 3)^2$. So, $L^{(4)}(k, x, \lambda) = 96(x(x + 3) + \lambda(1 - x)) < 0$ since $\lambda \geq x + 5$ and $x \geq 31$.

We now write $x = 31 + \zeta$ and $\lambda = 36 + \eta + \zeta$, where $\zeta \geq 0$ and $0 \leq \eta \leq 9$. Then

$$\begin{aligned}
L^{(3)}(8x - 17, x, \lambda) &= -48((15\eta + 13)\zeta^2 + (877\eta + 683)\zeta + 2(6403\eta + 4425)), \\
L^{(2)}(8x - 17, x, \lambda) &= -8((337\eta+242)\zeta^3+(29288\eta+18841)\zeta^2+(847963\eta+481289)\zeta+ \\
&\quad 6(1363153\eta + 669085)), \\
L^{(1)}(8x - 17, x, \lambda) &= -4((1680\eta + 929)\zeta^4 + 2(96871\eta + 46743)\zeta^3 + (8375014\eta + \\
&\quad 3439997)\zeta^2 + 20(8041770\eta + 2714417)\zeta + 20(57889062\eta + \\
&\quad 15214505)), \\
L(8x - 17, x, \lambda) &= -((12544\eta + 4639)\zeta^5 + (1802687\eta + 540598)\zeta^4 + 5(20717722\eta + \\
&\quad 4731103)\zeta^3 + (2975229843\eta + 461640976)\zeta^2 + 260(164275123\eta + \\
&\quad 13174991)\zeta + 100(2451743279\eta + 8302524)).
\end{aligned}$$

Thus $L(k, x, \lambda) < 0$ for all $k \geq 8x - 17$. Consequently, $\Delta < 0$ and there is no design with $k \geq 8x - 17$, $\lambda \geq x + 5$ and $x \geq 31$.

We must now consider $2x + 1 \leq k \leq 8x - 18$. Let $M(k, x, \eta) = (x + 5 + \eta)(2k^2(x + 1) - x(x + 3)) - 2(k - 1)x(x + 3)(k + 25 + 9\eta)$. Hence $M^{(2)}(k, x, \eta) = 4(\eta x + 3x + \eta + 5) > 0$.

Writing $x = 31+\zeta$, we get $M(2x+1, x, \eta) = -(73+28\eta)\zeta^3 - (7019+2697\eta)\zeta^2 - (224682+86483\eta)\zeta - 2394616 - 923302\eta$ and $M(8x - 18, x, \eta) = -(1+16\eta)\zeta^3 - (1427+2027\eta)\zeta^2 - (83800+78451\eta)\zeta - 1253004 - 960042\eta$. Hence, $M(k, x, \eta) < 0$ for $2x+1 \leq k \leq 8x-18$. That is, $\dfrac{(x + 5 + \eta)(2k^2(x + 1) - x(x + 3))}{2(k - 1)x(x + 3)} < k + 25 + 9\eta$. Combining this with the inequality in Theorem 3.2, we get $k + 1 \leq r \leq k + 29 + 9\eta$.

Write $r = k + t$. Then $1 \leq t \leq 29 + 9\eta$. From Lemma 2.1 (vi), k is a factor of $r(\lambda - r)$. Hence, k is a factor of $t(x + 5 + \eta - t)$. Write $s = t(x + 5 + \eta - t)/k$. Since $x + 5 + \eta - t \geq x - 24 - 8\eta \geq x - 24 > 0$, $x + 5 + \eta - t \leq x + 14 < 2x$ and $k \geq 2x + 1$, $1 \leq s < t$. Note also that 3 is a factor of $r - k - \lambda + x = t - \eta - 5$

Substituting $k = t(x + 5 + \eta - t)/s$, $r = k + t$, $\lambda = x + 5 + \eta$ and $y = x + 3$ into the quadratic equation for r in Lemma 2.4, we get a polynomial equation in x whose coefficients involve η, s and t. We examine each of these for $\eta = 0, \ldots, 9$, $t = 1, \ldots, 29+9\eta$ and $s = 1, \ldots, t - 1$, and find six cases in which there is an integer solution $x \geq 31$ with

x	v	k	λ	r	b	η	t	s
36	253	99	42	108	276	1	9	3
39	1065	210	44	224	1136	0	14	2
45	231	105	52	115	253	2	10	4
45	441	144	52	160	490	2	16	4
51	301	126	60	144	344	4	18	6
57	246	120	68	140	287	6	20	8

Table 1: parameters of the six possible exceptional designs in Theorem 3.3

$t-\eta-1$ divisible by 3 and such that the resulting values of v, k, λ, r and b are positive integers and the Calderbank criteria (Theorems 2.5 and 2.6) are satisfied. These are listed in Table 1. $\square$

We now consider the cases $2 \leq x \leq 30$.

Theorem 3.4. *Let $\mathcal{D}$ be a proper quasi-symmetric design with standard parameters with $2 \leq x \leq 30$ and $y = x+3$. Then the set of parameters of $\mathcal{D}$ is one of the 43 sets listed on Tables 3 and 4 or (λ, x) is one of the "exceptional" pairs $(7,2)$, $(8,2)$, $(9,2)$, $(10,2)$, $(8,3)$, $(9,3)$, $(9,4)$ and $(10,5)$.*

Proof. Recall from the proof of Theorem 3.1 that $\Delta = \lambda(\lambda F(k,x) + G(k,x))$, where $F(k,x) = 4k^4(1-x) + 8k^3x(x+3) - 4k^2x(x+2)(x+3) + x^2(x+3)^2$ and $G(k,x) = 4k(k-1)(k-x)(k-x-3)x(x+3)$. Clearly $G(k,x) > 0$ when $k > x+3$. As the coefficient of k^4 in $F(k,x)$ is negative, there is some integer k_x such that $F(k,x) < 0$ when $k \geq k_x$. For any $k \geq k_x$, $\lambda F(k,x) + G(k,x)$ is a decreasing function of λ. Hence we can find an integer λ_x such that $\Delta < 0$ when $k \geq k_x$ and $\lambda \geq \lambda_x$. Hence, we must have either $k < k_x$ or $\lambda < \lambda_x$.

Writing $x = 2+\zeta$ and $k = 5x+8+\xi$, we get $F(k,x) = -4(\zeta+1)\xi^4 - 8(9\zeta^2+39\zeta+26)\xi^3 - 4(121\zeta^3+923\zeta^2+2006\zeta+904)\xi^2 - 16(5\zeta+18)(18\zeta^3+131\zeta^2+259\zeta+74)\xi - (1600\zeta^5+22519\zeta^4+118922\zeta^3+279691\zeta^2+248980\zeta+5084)$ and $G(k,x) = 4(\zeta+2)(\zeta+5)(\xi^4+2(9\zeta+32)\xi^3+(121\zeta^2+861\zeta+1529)\xi^2+(360\zeta^3+3845\zeta^2+13663\zeta+16154)\xi+4(\zeta+4)(4\zeta+13)(5\zeta+17)(5\zeta+18))$. Hence, we may take $k_x = 5x+8$.

Writing $\lambda = x+500+\eta$, we get $\Delta = -4(\eta\zeta+496\zeta+\eta+492)\xi^4 - 8(9\eta\zeta^2+4462\zeta^2+39\eta\zeta+19290\zeta+26\eta+12732)\xi^3 - 4(121\eta\zeta^3+59957\zeta^3+923\eta\zeta^2+456586\zeta^2+2006\eta\zeta+988603\zeta+904\eta+438518)\xi^2 - 4(360\eta\zeta^4+178271\zeta^4+3916\eta\zeta^3+1936266\zeta^3+14612\eta\zeta^2+7205107\zeta^2+20128\eta\zeta+9859876\zeta+5328\eta+2513116)\xi - (1600\eta\zeta^5+791719\zeta^5+22519\eta\zeta^4+11126276\zeta^4+118922\eta\zeta^3+58611863\zeta^3+279691\eta\zeta^2+137170342\zeta^2+248980\eta\zeta+120335060\zeta+5084\eta+

x	λ_x	k_x	x	λ_x	k_x	x	λ_x	k_x	x	λ_x	k_x	x	λ_x	k_x
2	501	18	3	86	16	4	1081	15	5	148	16	6	117	17
7	116	18	8	128	19	9	149	20	10	184	21	11	81	23
12	59	25	13	50	27	14	45	29	15	43	31	16	41	33
17	40	35	18	40	37	19	40	39	20	40	41	21	40	43
22	40	45	23	41	47	24	41	49	25	42	51	26	43	53
27	43	55	28	44	57	29	45	59	30	46	61			

Table 2: smaller values for λ_x and k_x in Theorem 3.4

x	v	k	λ	r	b
2	71	14	39	210	1065
3	39	12	22	76	247
3	45	15	42	132	396
3	55	15	63	243	891

x	v	k	λ	r	b
3	57	15	30	120	456
3	60	15	14	59	236
6	33	15	35	80	176

Table 3: parameters of the possible designs in Theorem 3.4 with $k \leq k_x$.

6248). So, we may take $\lambda_x = x + 500$. In most cases, we can find smaller values for λ_x and k_x than are given by these general formulas. Table 2 contains such smaller values for λ_x and k_x.

We now consider all values of k satisfying $k \geq 2x+1$, $k \geq x+4$ and $k \leq k_x$ and all values of λ satisfying $\lambda \geq x+1$, $\lambda \leq k(k-1)/2$ and $\lambda \leq -G(k,x)/F(k,x)$, if $F(k,x) < 0$. For each such choice we determine Δ and, if it is a perfect square, determine the two corresponding rational numbers $\dfrac{-b_1 \pm \sqrt{\Delta}}{2a_1}$ which are possible values for r. Corresponding value for v and b are then determined. There are ten such parameter sets meeting all the basic integrality requirements, $v \geq 2k$ and the Calderbank requirements. Of these ten, three satisfy the equation $b = v(v-1)/2$. From [26, Theorems 3.16 and 9.5], there are no designs corresponding to these parameter sets. The remaining eight parameter sets are listed in Table 3.

Next, we take a fixed x satisfying $2 \leq x \leq 30$ and a fixed λ satisfying $x + 5 \leq \lambda \leq \lambda_x$. We show that for most such pairs (λ, x) we can put an upper bound on the value of k for which a design exists. Again, we do this by showing that $\Delta < 0$ for sufficiently large k.

In this case, $\lambda F(k,x) + G(k,x) = -4((\lambda - x - 4)(x-1) - 4)k^4 + 8x(x+3)(\lambda - x - 2)k^3 - 4x(x+3)((x+2)(\lambda - x - 5) + (2x+7))k^2 - 4x^2(x+3)^2k + \lambda x^2(x+3)^2$. If $(\lambda - x - 4)(x-1) > 4$ then we can find $k_{\lambda,x}$ such that $\lambda F(k,x) + G(k,x) < 0$, and hence $\Delta < 0$, when $k \geq k_{\lambda,x}$. We now consider the pairs (λ, x) for which $(\lambda - x - 4)(x-1) > 4$ and $2 \leq x \leq 30$; in detail, these are $\lambda = x+5$ and $x \geq 6$, $\lambda = x+6$ and $x \geq 5$, $\lambda = x+7$

x	v	k	λ	r	b
2	101	20	19	100	505
2	110	20	38	218	1199
2	115	20	54	324	1863
2	116	20	57	345	2001
2	125	20	57	372	2325
2	134	20	30	210	1407
2	161	23	33	240	1680
3	63	18	17	62	217
3	69	18	30	120	460
3	72	18	34	142	568
3	91	21	18	81	351
3	101	21	21	105	505
3	111	21	14	77	407
3	144	27	12	66	352
4	43	16	40	112	301
4	46	16	72	216	621
4	49	16	45	144	441
5	56	20	19	55	154

x	v	k	λ	r	b
5	65	20	19	64	208
6	42	18	51	123	287
6	45	18	34	88	220
6	57	21	25	70	190
6	61	21	21	63	183
6	171	36	14	68	323
8	41	20	57	120	246
9	42	21	60	123	246
9	177	45	15	60	236
12	66	30	29	65	143
12	77	33	24	57	133
12	99	36	20	56	154
12	144	45	20	65	208
12	155	45	18	63	217
12	1065	120	17	152	1349
15	78	36	30	66	143
15	405	81	20	101	505
28	325	100	33	108	351

Table 4: parameters of the possible designs in Theorem 3.4 with $k \le k_{\lambda,x}$, excluding Table 3 entries.

and $x \ge 4$, $\lambda = x+8$ and $x \ge 3$, $\lambda \ge x+9$ and $x \ge 2$. For each such pair we determine a suitable $k_{\lambda,x}$. The following choices for $k_{\lambda,x}$ are appropriate: $k_{x+5,x} = 6x+309$ when $x \ge 6$, $k_{x+6,x} = 4x+109$ when $x \ge 4$, $k_{x+7,x} = 3x+81$ when $x \ge 3$, $k_{x+8,x} = 3x+44$ when $x \ge 3$, and $k_{\lambda,x} = 3x+133$ when $x \ge 2$ and $x+9 \le \lambda \le \lambda_x$. For each such pair (λ, x), we determine all possible parameter sets with $2x+1 \le k \le k_{\lambda,x}$ meeting all the basic integrality requirements, $v \ge 2k$ and the Calderbank requirements. There are 39 such sets, excluding those that have appeared already in Table 3. Three of these fail since some parameters of the corresponding strongly regular graph are not integers. The remaining 36 are listed in Table 4.

The pairs (λ, x), where $2 \le x \le 30$ and λ satisfying $x+5 \le \lambda \le \lambda_x$, which failed the inequality $(\lambda - x - 4)(x-1) > 4$ are $(\lambda, x) = (7,2)$, $(8,2)$, $(9,2)$, $(10,2)$, $(8,3)$, $(9,3)$, $(9,4)$ and $(10,5)$. $\square$

We are unable to determine if the number of standard parameter sets corresponding to the exceptional pairs in Theorem 3.4 is finite, although computational investigations suggest that such sets are rare. For $(8,2)$ and $(9,4)$, we found parameter sets which passed all our tests; namely, $(1001, 65, 8, 125, 1925)$ with $x = 2$ and $(4642, 154, 9, 273, 8229)$ with $x = 4$.

Similarly, computations for each of the four cases $\lambda = x+1, \dots, \lambda = x+4$, with $x \ge 0$, suggest that the number of standard parameter sets is rare, but we have been unable to show that this number is finite in any of these cases. In each case, we found at least one standard parameter set which passed all our tests.

We end with an observation concerning certain quasi-symmetric designs whose intersection numbers are 0 and 3. Lam and Tonchev [10] classified all affine $2-(27,9,4)$ designs. We show below that their result classifies all quasi-symmetric designs with these parameters. We need the following theorem of Majumdar [12].

Theorem 3.5 (Majumdar)**.** *Let A and B be distinct blocks of a 2-(v,k,λ) design $\mathcal{D}$ of order $n = r - \lambda$. Then $|A \cap B| \geq k - n$. Moreover, if $|A \cap B| = k - n$, then for any block C of $\mathcal{D}$, other than A or B, $|C \cap A| = |C \cap B|$.*

Proposition 3.6. *Let $\mathcal{D}$ be a quasi-symmetric $2-(27,9,4)$design. Then $x = 0$ and $y = 3$ and $\mathcal{D}$ is an affine resolvable design.*

Proof. The parameters of $\mathcal{D}$ are $v = 27$, $b = 39$, $r = 13$, $k = 9$, $\lambda = 4$. Using Lemma 2.3, part (iii), $x < 9^2/27 = 3$. So $x = 0, 1,$ or 2. From Lemma 2.1, part(v), we get $19xy - 54(x+y) + 162 = 0$. Hence, $y = 3$, $108/35$ or $27/8$ according as $x = 0$, 1 or 2.

So $\mathcal{D}$ has intersection numbers $x = 0$ and $y = 3$. From Lemma 2.1, part (iii), $a = 36$. That is, each block of $\mathcal{D}$ meets 36 blocks in 3 points. Since $r = 13 = k + \lambda$, if A and B are disjoint blocks Majumdar's Theorem 3.5 implies that every other block is either disjoint from both A and B or meets each of A and B in 3 points. Simple counting arguments show that (i) given any two disjoint blocks A and B, there is a unique block disjoint from A and B and (ii) any point not on a given block A lies on a unique block B which is disjoint from A. It follows that $\mathcal{D}$ is resolvable and in fact affine. $\square$

4 Triangle-free quasi-symmetric designs

In this last section, some results on triangle-free quasi-symmetric designs are given. This is a topic which is of current interest. In a recent preprint, Klin and Woldar[9] remark in their paper "A further hope is that our text will help to promote future investigations of such extremely rare objects as primitive strongly regular graphs with no triangles". Recall that a strongly regular graph Γ is *primitive*, if both Γ and its complement $\overline{\Gamma}$ are connected. Both the block graph of a quasi-symmetric design and the complement of this graph are strongly regular. So quasi-symmetric designs whose block graph and its complement are connected give rise to primitive strongly regular graphs. So earlier papers, such as [1], [25], [18], [11], [19], [20], [21], [22], and [23] may be viewed under this wider umbrella.

Pawale [21], proved the following result:

Theorem 4.1. *Let $\mathcal{D}$ be a proper quasi-symmetric with parameters $(v, b, r, k, \lambda; x, y)$ with $x \neq 0$ and $z = y - x = 1$. Then $\mathcal{D}$ is a design with parameters in (1), (2) as follows or $\mathcal{D}$ is a complement of one of the designs in (1).*

(1): $v = (m+1)(m+2)/2,\ b = (m+2)(m+3)/2,\ r = m+3,\ k = m+1,\ \lambda = 2,$ $x = 1,\ y = 2,$ *and* $m = 2, 3, \ldots.$

(2): $v = 5,\ b = 10,\ r = 6,\ k = 3,\ \lambda = 3,$ *and* $x = 1,\ y = 2.$

Note that designs in (1) are residuals of biplanes and the design in (2) is an embeddable trivial design.

Triangle free quasi-symmetric designs (i.e. those for which the block graph has no triangles, or equivalently the design has no three distinct blocks such that any two of them intersect in x points), with $x = 0$ were first studied by Baartmans and Shrikhande [1], and then by Limaye, Sane, and Shrikhande [11]. Shrikhande [25] and Pawale [20] investigated triangle-free quasi-symmetric designs with $x \geq 1$. In [25, Lemma 2.5] it was asserted that for such quasi-symmetric designs λ satisfies a quadratic equation of the form $A\lambda^2 + B\lambda + C = 0$, where A, B, and C are certain polynomial functions in k with coefficients involving x and y. Using this lemma, it was concluded in [25, Theorem 2.9] that if y is larger than a certain function of x for such quasi-symmetric designs then there are only finitely many such designs. However, the expressions for A, B, and C had some errors, which were corrected by Pawale in [20, Lemma 3.1] where he obtained a stronger result than that claimed in [25, Theorem 2.9].

Lemma 4.2 ([20, Lemma 3.1])**.** *Let $\mathcal{D}$ be a triangle-free quasi-symmetric design with standard parameter set $(v, b, r, k, \lambda; x, y)$. Then $A\lambda^2 + B\lambda + C = 0$, where*

$$\begin{aligned}
A &= k\big(k^3y - x^2(x^2 - xy + y^2) + k(x^3 + 2x^2y - xy^2 + y^3) + k^2(x^3 - y^2(1+y) \\
&\quad +xy(-1+3y) - x^2(1+3y))\big); \\
B &= -2k^4y^2 + x^3y(-2x+y) - k^2(2x^4 - 5xy^3 - x^3(1+y) + y^3(1+y) \\
&\quad +3x^2y(1+3y)) + kx(-4x^2y^2 + 3xy^2(1+y) - y^3(1+y) + x^3(1+4y)) \\
&\quad +k^3y(-2x^3 + x^2(3+6y) + x(-1+3y-6y^2) + 2(y+y^3)); \\
C &= (-1+k)(k-y)y(-2x^2 + ky + xy)^2.
\end{aligned}$$

Theorem 4.3 ([20, Theorem 3.2])**.** *For a fixed positive integer y, there exist only finitely many quasi-symmetric triangle-free designs with the larger intersection number y.*

Meyerowitz, Sane, and Shrikhande[18] investigated quasi-symmetric designs using MACSYMA.

Triangle-free quasi-symmetric 3-designs are completely classified in Pawale [19]:

Theorem 4.4. *Let $\mathcal{D}$ be a quasi-symmetric 3-design with intersection numbers x and y $(0 \le x < y < k)$. Then $\mathcal{D}$ is triangle-free if and only if $\mathcal{D}$ is a Hadamard 3-design or $\mathcal{D}$ has parameter set (v, k, λ), where $v = (\lambda+2)(\lambda^2+4\lambda+2)+1$ and $k = \lambda^2+3\lambda+2$, or $\mathcal{D}$ is a complement of one of these designs.*

Pawale [20] characterized triangle-free quasi-symmetric 2-designs with $x \neq 0$ and $k = 2y - x$ as being a trivial design with $v = 5$ and $k = 3$. He also showed that triangle-free quasi-symmetric designs with $\lambda = y$, or $\lambda = y - 1$ do not exist.

Pawale also remarks that there is strong reason to believe that triangle-free quasi-symmetric designs with $x \neq 0$ do not exist. In support of this belief, Pawale in [21], proves that triangle-free quasi-symmetric designs with $y - x = 2$ do not exist and established the following bound when $y - x = z \geq 1$.

Theorem 4.5 ([21, Theorem 4.1])**.** *Let $\mathcal{D}$ be a triangle-free quasi-symmetric design with standard parameter set $(v, b, r, k, \lambda; x, y)$ and $v \geq 2k$. Let $z = y - x$. Then $x \leq z + z^2$.*

Theorem 4.6. *There are no proper triangle-free quasi-symmetric designs with non-zero intersection numbers x and y with $y = x + 3$.*

Proof. Let $\mathcal{D}$ be a triangle-free quasi-symmetric design with standard parameter set $(v, b, r, k, \lambda; x, x+3)$. We may assume $v \geq 2k$. By Lemma 2.3(iv), $k > 2x$. By Theorem 4.5, $x \leq 12$.

By Lemma 4.2, λ is a solution of the quadratic $A\lambda^2 + B\lambda + C = 0$, where A, B and C are as described in that lemma. Since A, B and C are integers, the discriminant $D_0 = B^2 - 4AC$ is the square of an integer. However, $D_0 = DE^2$ where

$$\begin{aligned} D &= -k^5(12x+36) + k^4(37x^2+120x+468) - k^3(40x^3+144x^2+90x+432) \\ &\quad +k^2(18x^4+72x^3-54x^2-324x+81) - k(4x^5+12x^4-18x^3-108x^2-162x) \\ &\quad +x^6-18x^4+81x^2 \end{aligned}$$

and $E = x^2 + 2kx + 6k$.

When $k = x + 11 + \zeta$, $D = -(12x+36)\zeta^5 - (23x^2+720x+1512)\zeta^4 - (12x^3+1036x^2+15378x+23400)\zeta^3 - (396x^3+15006x^2+146094x+153567)\zeta^2 - (2844x^3+78352x^2+586350x+298782)\zeta - 12x^2(12-x)(27x+325) - (41195x^2+631686(x-1)+142725)$ and this is clearly negative when $1 \leq x \leq 12$ and $\zeta \geq 0$. Recall that $k > y = x + 3$. The remaining cases $1 \leq x \leq 12$, $\max(2x+1, x+4) \leq k \leq x + 10$ are now examined one by one. In a few cases $D < 0$, e.g. $D = -187964$ when $(x, k) = (4, 14)$. In most cases D is a non-square positive integer, e.g. $\sqrt{D} = 168\sqrt{22}$ when $(x, k) = (3, 7)$. In the remaining cases $(x, k) = (1, 7)$, $(1, 8)$, $(2, 8)$, $(3, 13)$, $(4, 10)$ and $(5, 11)$, the roots of the quadratic equation $A\lambda^2 + B\lambda + C = 0$ are $\{-25/11, 48/7\}$, $\{578/83, -4\}$, $\{35/4, -7/2\}$, $\{624/37, 63/4\}$, $\{-121/23, 63/5\}$, and $\{-169/29, 160/11\}$, respectively. Consequently, $\mathcal{D}$ does not exist. $\square$

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